Microsoft Research

Packing Multicast Trees

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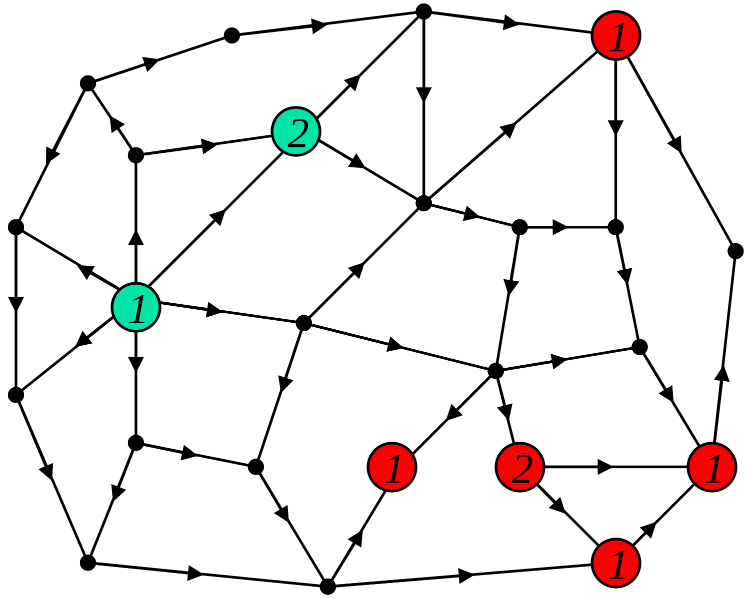
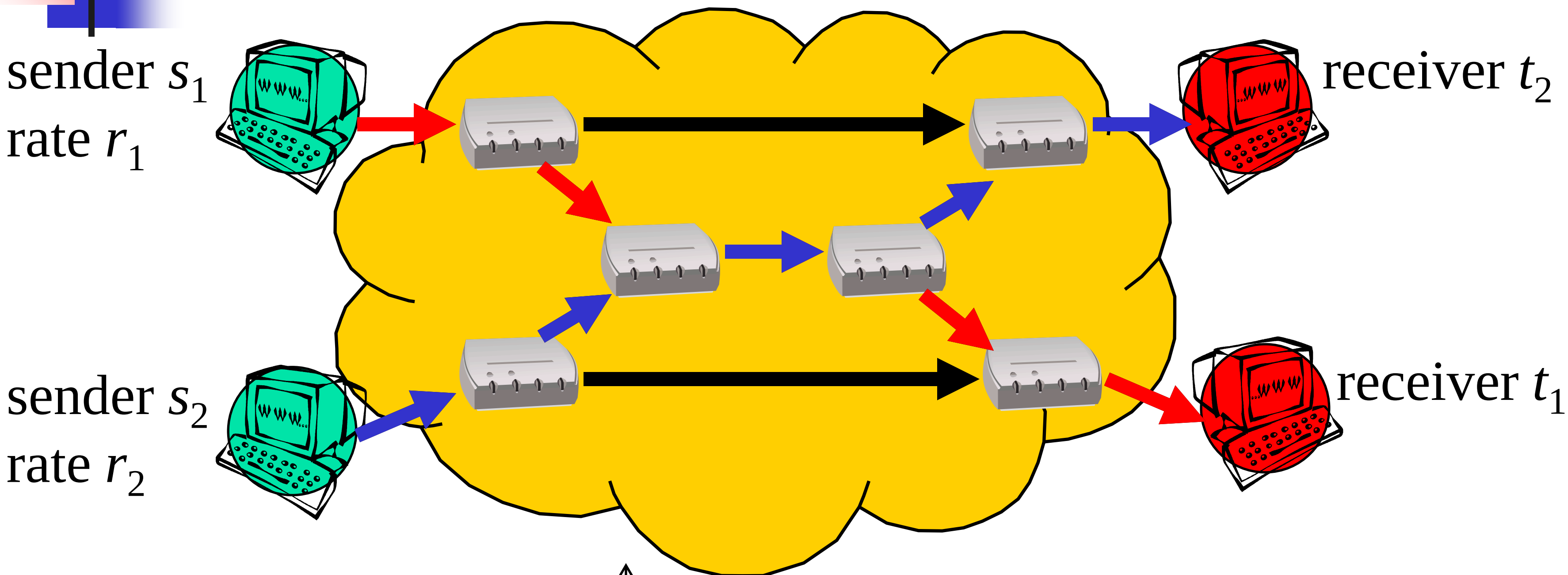
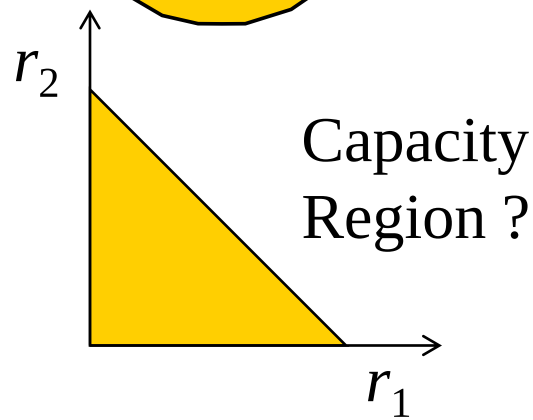
Problem



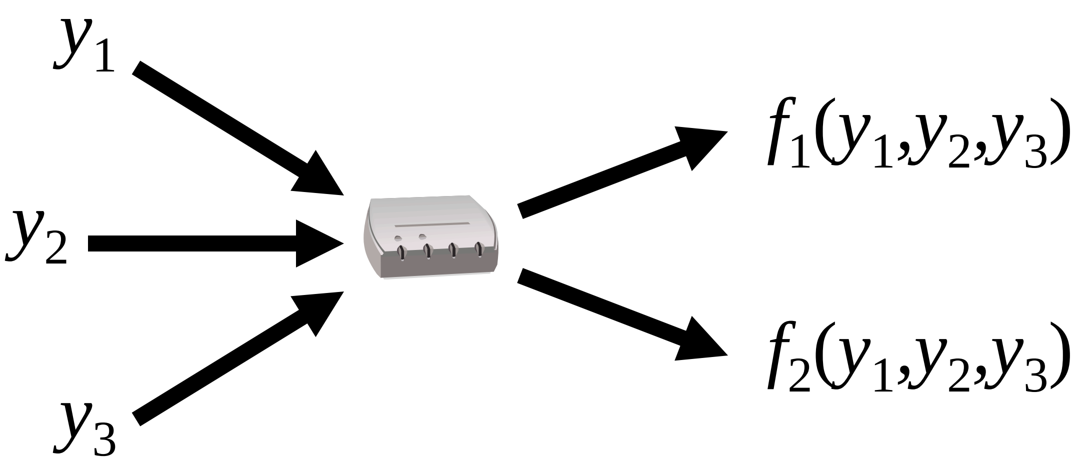
- Given
  - Directed graph (V,E) with edge capacities $c(e),\ e \in E$
  - Multiple multicast sessions $\{(s_i,T_i)\}$, each with
    - Sender s_i (i)
    - Receiver set T_i (i)
    - Transmission rate r_i
- What is the capacity region $\mathsf{R} = \{\text{achievable } (r_1,\ldots,r_{|S|})\}$?
- What is a transmission scheme that achieves a given transmission rate vector $(r_1,\ldots,r_{|S|}) \in \mathsf{R}$?

Butterfly Network with Two Unicast Sessions





Network Coding vs Routing



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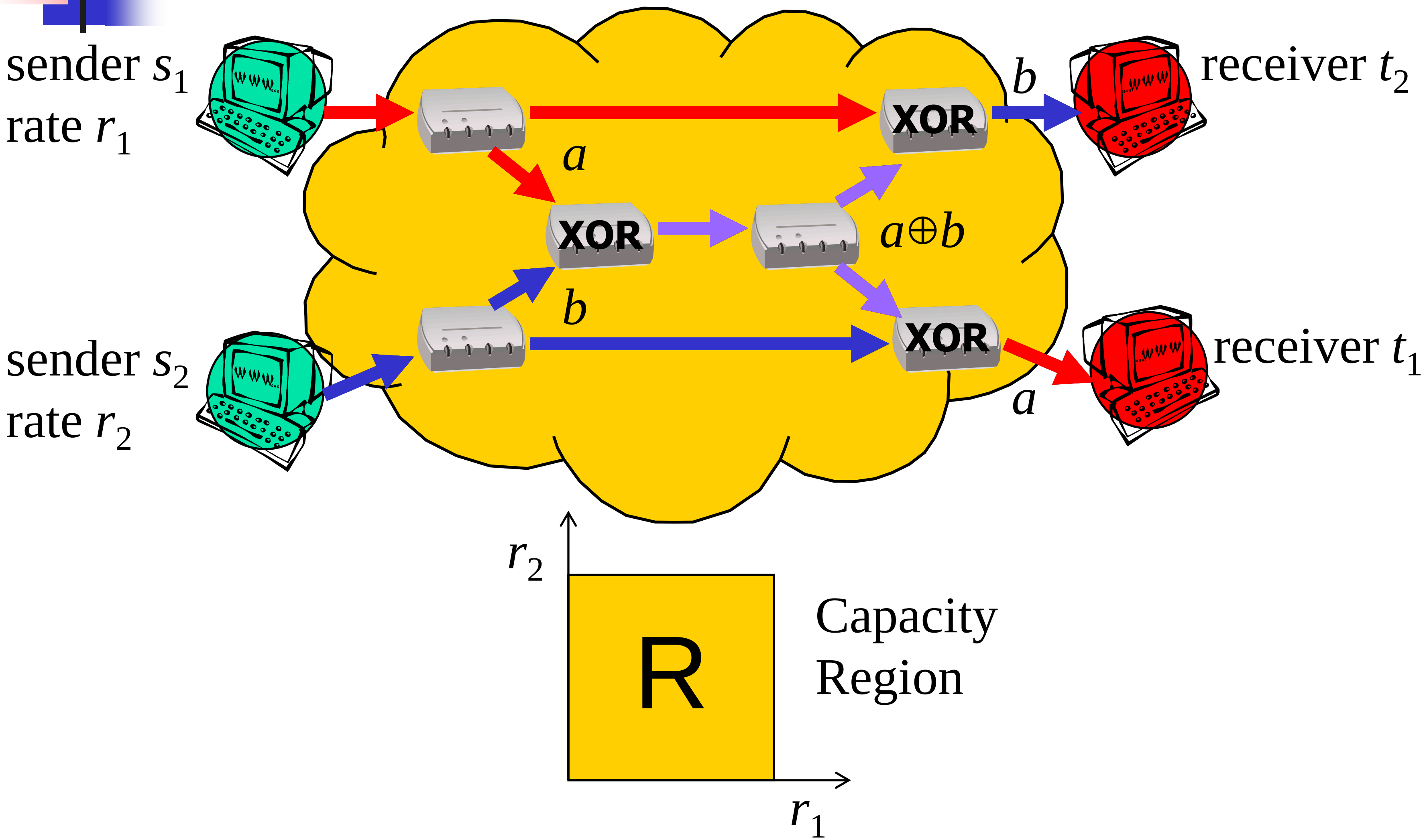


Characterization of Capacity for Acyclic Graphs

$$\mathrm{N} = \{Y_s : s \in S\} \cup \{U_e : e \in E\}$$ -- N random variables

$$\Gamma^*_{\mathrm{N}} = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : \mathbf{h} \text{ is entropic} \right\}$$

$$C_1 = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : h_{Y_S} = \sum_{s \in S} h_{Y_s} \right\}$$

$$C_2 = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : h_{U_{Out(s)}|Y_s} = 0, \forall s \in S \right\}$$

$$C_3 = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : h_{U_{Out(v)}|U_{In(v)}} = 0, \forall v \in V \setminus (S \cup T) \right\}$$

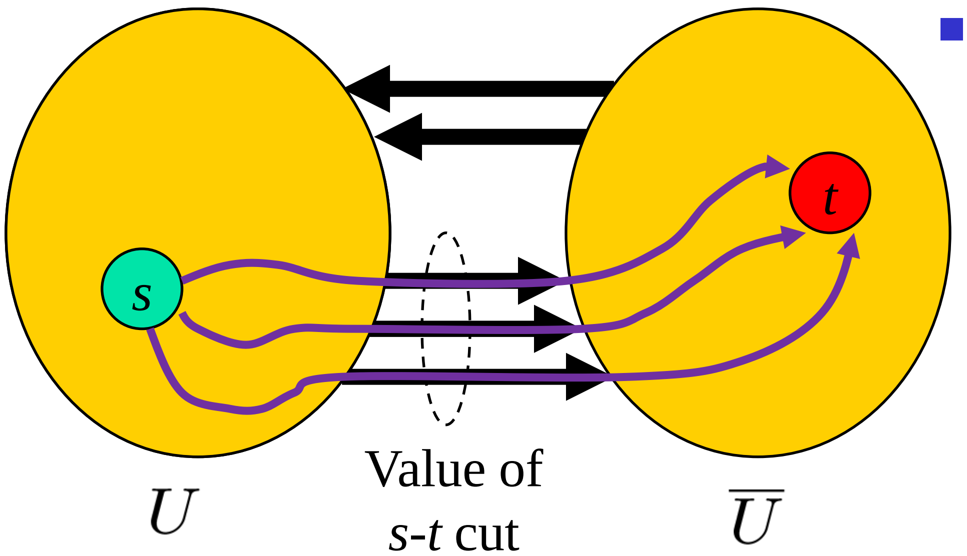
$$C_4 = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : h_{U_e} \le c(e), \forall e \in E \right\}$$

$$C_5 = \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} : h_{Y_{S_t}|U_{In(t)}} = 0, \forall t \in T \right\}$$

$$\mathrm{R} = \Lambda\left(proj_{Y_S}\left(\overline{con\left(\Gamma^*_{\mathrm{N}} \cap C_1 \cap C_2 \cap C_3\right)} \cap C_4 \cap C_5 \right)\right)$$

[Yan, Yeung, Zhang; ISIT 2007]

Single Unicast Session

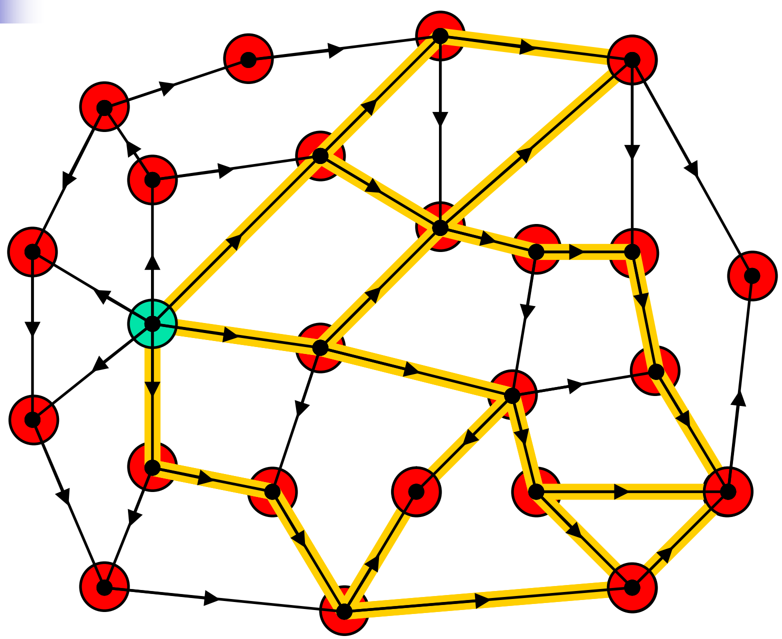


- Given
  - Directed graph (V,E) with edge capacities $c(e),\ e \in E$
  - Single unicast session with
    - Sender s
    - Receiver t
    - Transmission rate r

- $r \le$ MinCut(s,t)
- MinCut(s,t) is achievable, i.e., MaxFlow(s,t) = MinCut(s,t), by packing edge-disjoint directed paths

[Menger; 1927]

Single Broadcast Session

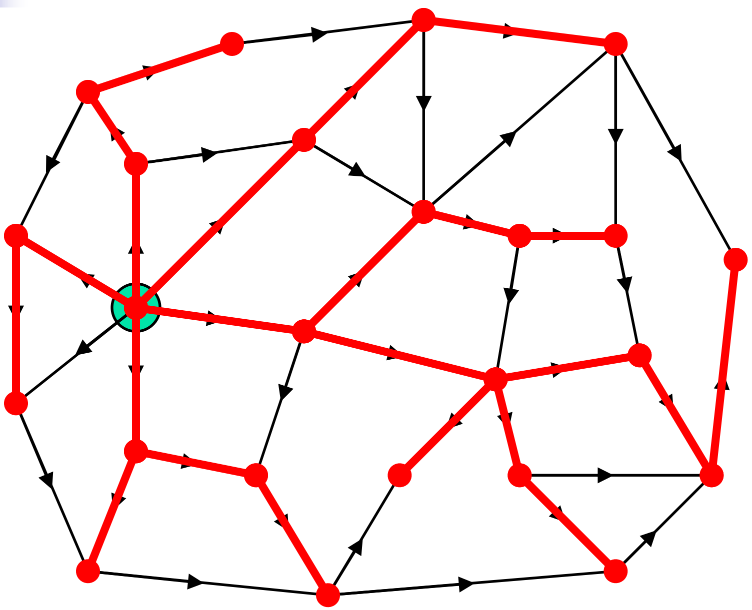


- Given
  - Directed graph (V,E) with edge capacities $c(e),\ e \in E$
  - Single broadcast session with
    - Sender s
    - Receiver set $T=V$
    - Transmission rate r

- $r \leq \min_{v \in V} \text{MinCut}(s,v)$
- $\min_{v \in V} \text{MinCut}(s,v)$ is achievable ("broadcast capacity") by packing edge-disjoint directed spanning trees

[Edmonds; 1972]

Single Broadcast Session



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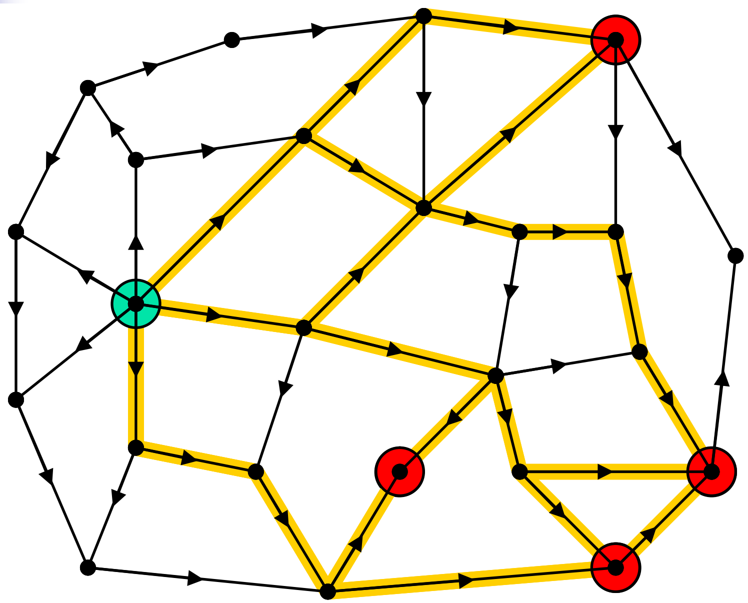
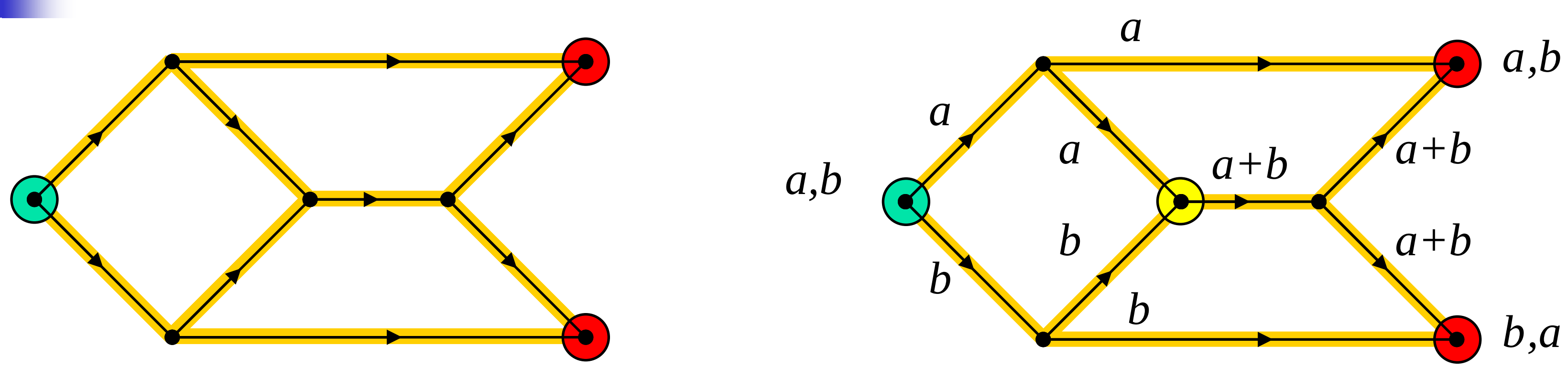
[Edmonds; 1972]

Single Multicast Session



- Given
  - Directed graph (V,E) with edge capacities $c(e),\ e \in E$
  - Single multicast session with
    - Sender s
    - Receiver set $T \subset V$
    - Transmission rate r
- $r \leq \min_{t \in T} \text{MinCut}(s,t)$
- $\min_{t \in T} \text{MinCut}(s,t)$ is NOT always achievable by packing edge-disjoint multicast (Steiner) trees

Packing Multicast Trees Insufficient to achieve MinCut



optimal routing
throughput = 1

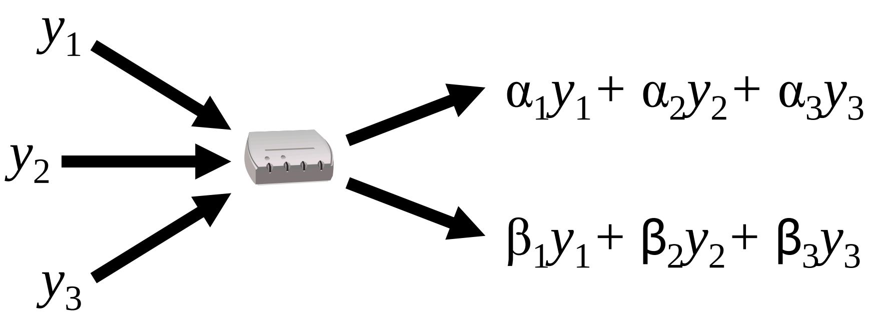
network coding
throughput = 2

- $\min_{t \in T} \mathrm{MinCut}(s,t)$ is always achievable by network coding

[Alswede, Cai, Li, Yeung ; 2000]

Linear Network Coding Sufficient to achieve MinCut

- Linear network coding is sufficient to achieve $\min_{t \in T} \text{MinCut}(s,t)$



[Cai, Li, Yeung; 2003]
[Koetter and Médard; 2003]

- Polynomial time algorithm for finding coefficients

[Jaggi, Chou, Jain, Effros; Sanders, et al.; 2003]
[Erez, Feder; 2005]

Linear Network Coding Sufficient to achieve MinCut

- Packing a maximum-rate set of multicast trees is NP hard

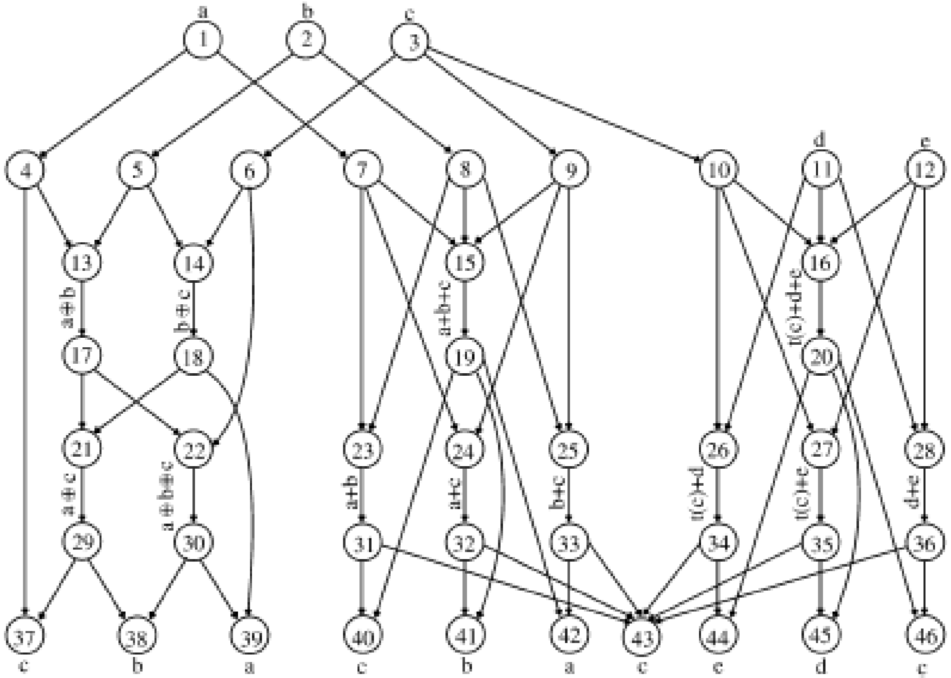
  [Jain, Mahdian, Salavatipour; 2003]

- Rate gap to $\min_{t \in T} \text{MinCut}(s,t)$ can be a factor of $\log n$

  [Jaggi, Chou, Jain, Effros; Sanders, et al.; 2003]

Linear Network Coding NOT Sufficient for Multiple Sessions

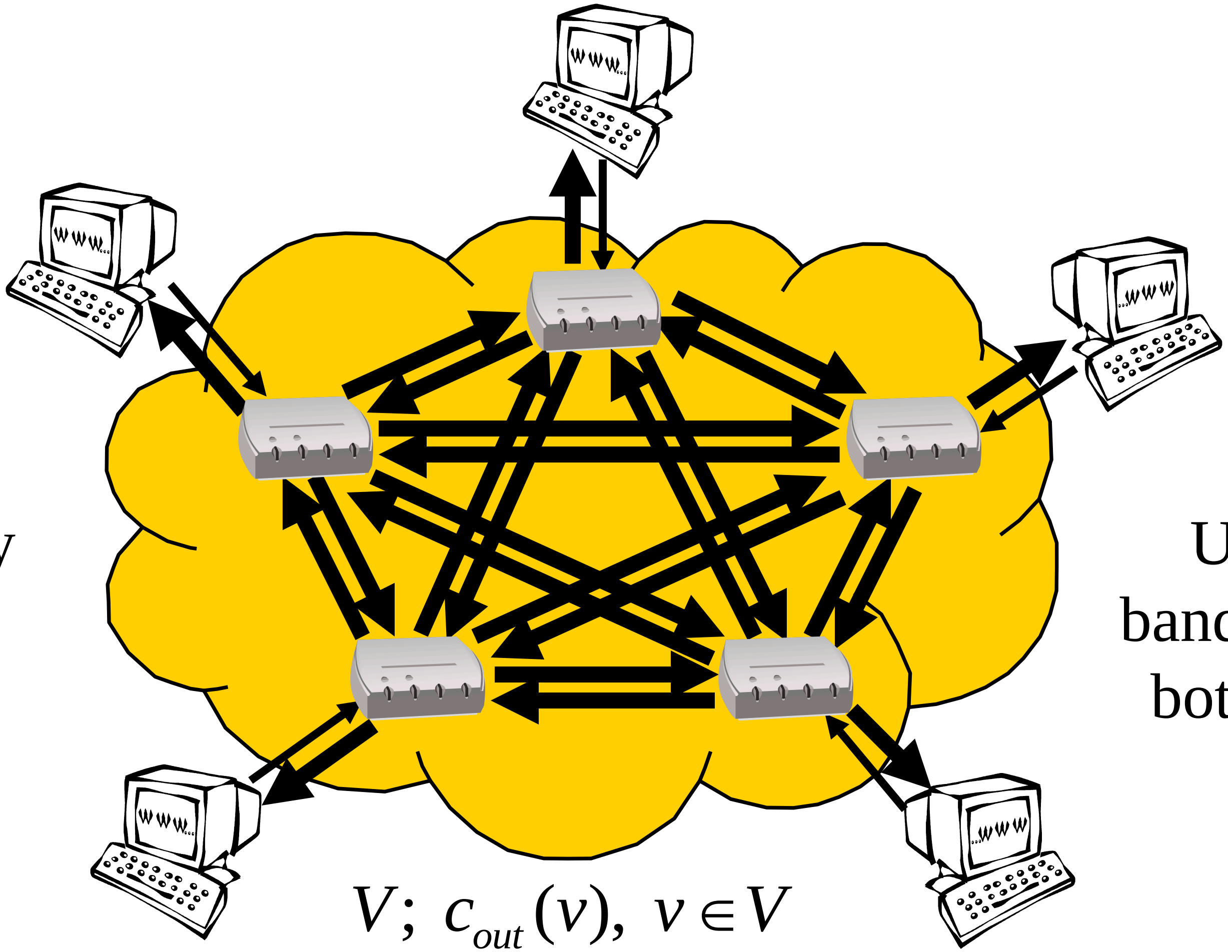
- Linear network coding is NOT generally sufficient to achieve capacity for multiple sessions



[Dougherty, Freiling, Zeger; 2005]

P2P Networks

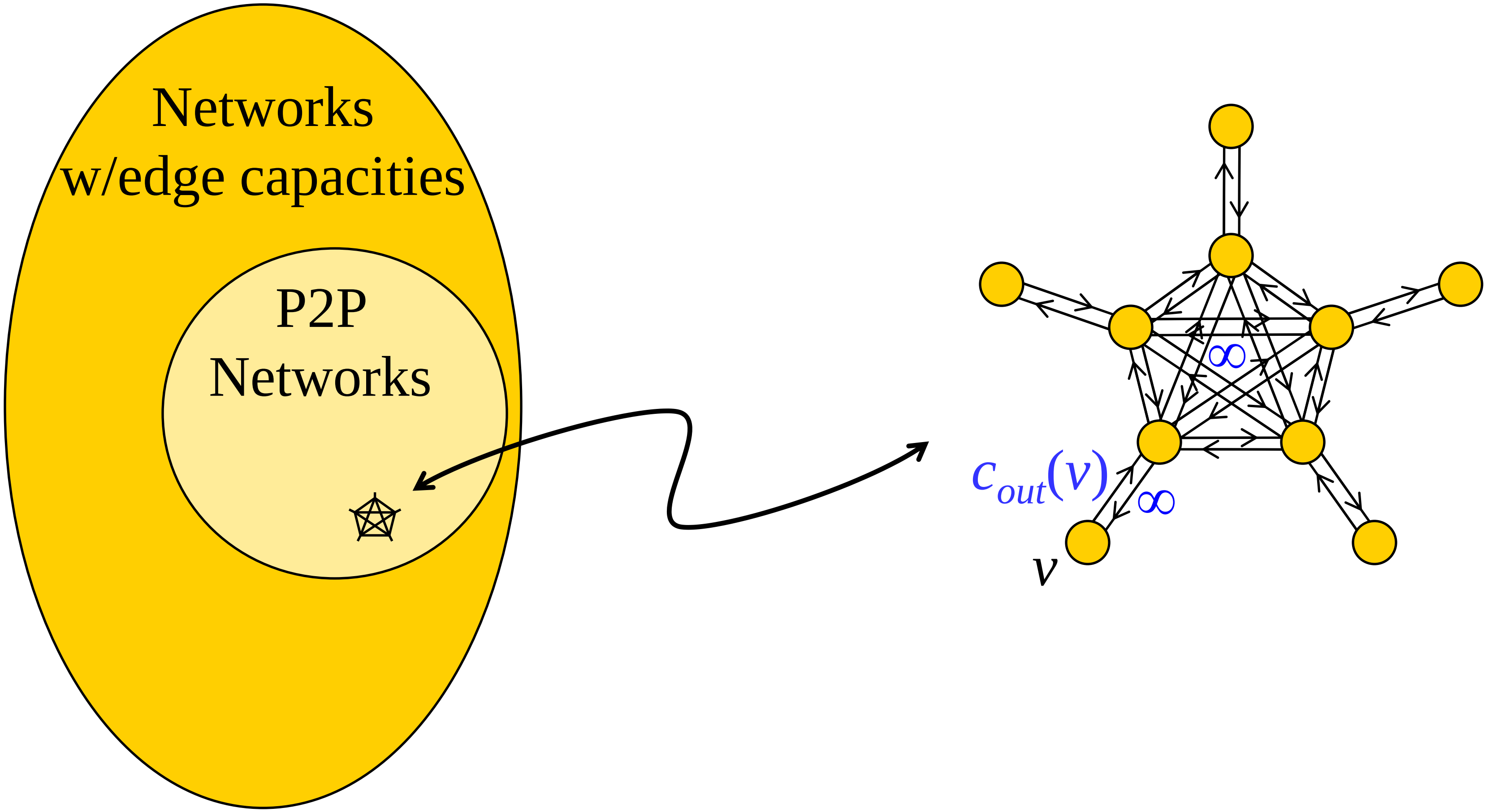
Completely connected overlay



$V;\ c_{out}(v),\ v \in V$

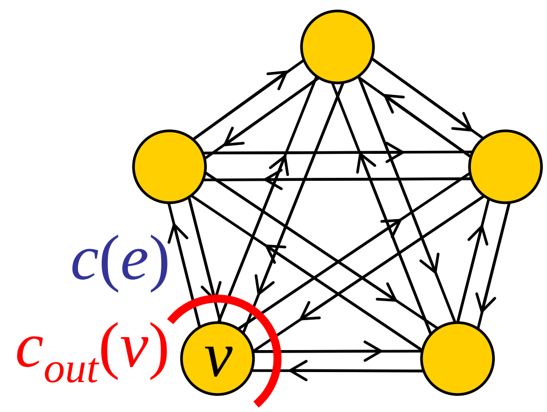
Upload bandwidth is bottleneck

P2P Network as Special Case



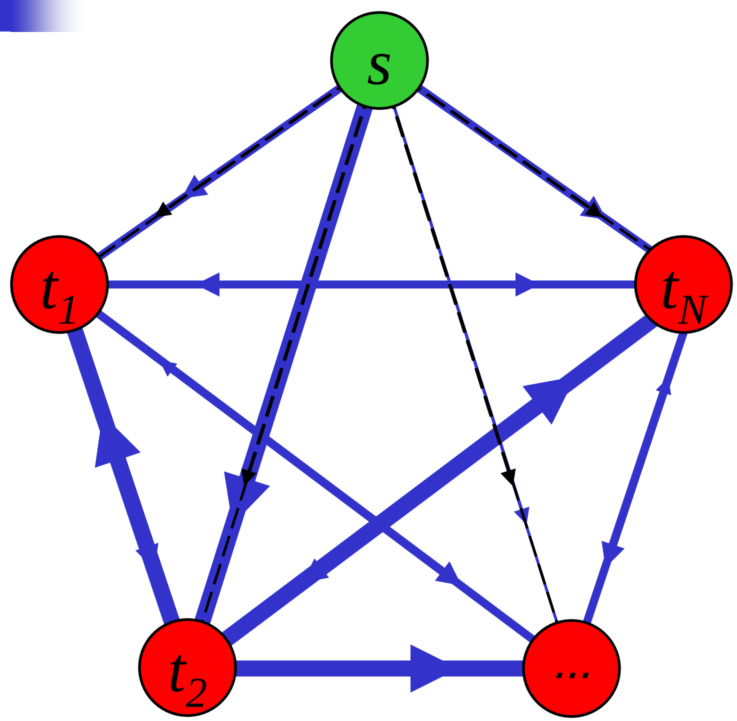
P2P Network as Set of Networks

$$\underbrace{\langle V;\ c_{out}(v),\ v \in V \rangle}_{\text{P2P network}} \Leftrightarrow \left\{ \underbrace{\langle (V,E); c(e), e \in E \rangle}_{\text{network w/edge capacities}} : \sum_{e \in Out(v)} c(e) \le c_{out}(v), v \in V \right\}$$



- Corollary to Edmonds' Theorem:
  - Given a P2P network with a single broadcast session (i.e., a single sender and all other nodes as receivers), the maximum throughput is achievable by routing over a set of directed spanning trees

Mutualcast



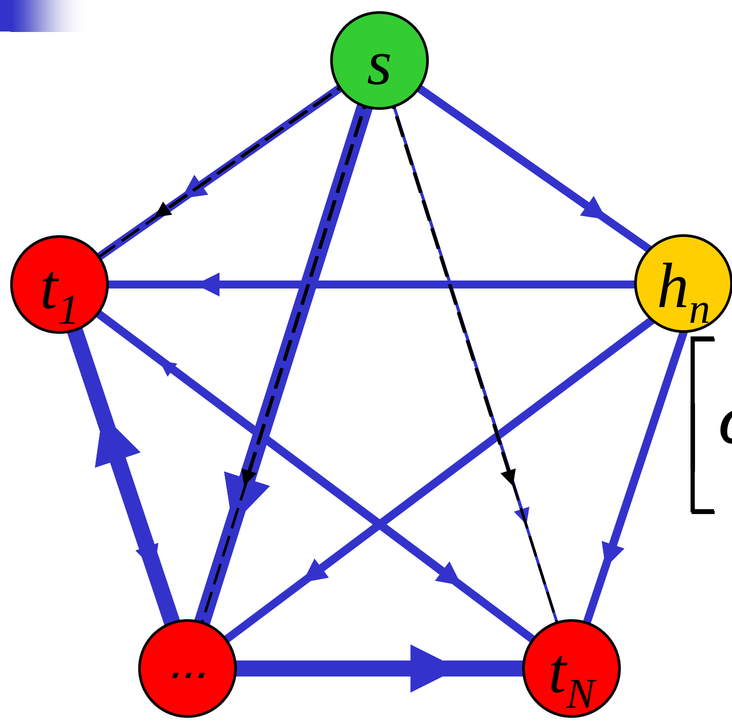
Throughput:

$$\left[c_{out}(s) - \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} \right] / N + \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1}$$

- Cutset analysis: Throughput achieves mincut bound

[Li, Chou, Zhang; 2005]

Mutualcast with Helpers



Throughput:

$$\left[c_{out}(s) - \sum_{n=1}^{N}\frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M}\frac{c_{out}(h_n)}{N-1}\right]/N$$

$$+\sum_{n=1}^{N}\frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M}\frac{c_{out}(h_n)}{N}$$

- Cutset analysis: Throughput achieves mincut bound

[Li, Chou, Zhang; 2005]

Mutualcast Theorem

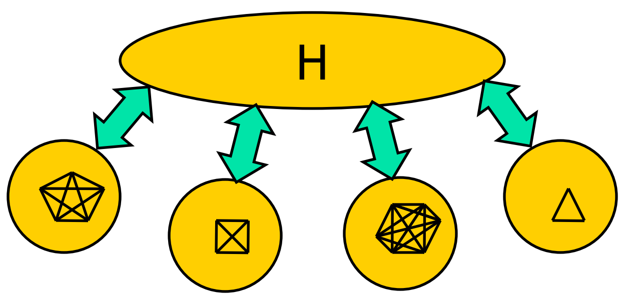
- Given
  - P2P network
  - Source s, receiver set T, helper set $H = V - T - s$
- Then any rate $r \le C$ can be achieved by routing over at most $|V|$ depth-1 and depth-2 multicast trees
- Capacity is given by

$$C = \min\left\{ c_{out}(s), \left[c_{out}(s) - \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} - \sum_{n=1}^{M} \frac{c_{out}(h_n)}{N} \right] / N + \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M} \frac{c_{out}(h_n)}{N} \right\}$$

- Remarks:
  - Despite existence of helpers, no network coding needed!
  - $|V|$ trees instead of $O(|V|^{|V|})$
  - Simple, short trees

Multi-session Mutualcast Theorem

- Given
  - P2P network
  - <u>Multiple</u> sources s_i and receiver sets T_i, $i=1,\ldots,|S|$, s.t. sets $(s_i \cup T_i)$ are identical or disjoint; no connections between the disjoint receiver sets; $H = V - \cup(s_i \cup T_i)$



- Then any rate vector $(r_1,\ldots,r_{|S|}) \in \mathrm{R}$ can be achieved by routing over at most $|V|\times|S|$ depth-1 and depth-2 multicast trees

[Sengupta, Chen, Chou, Li; ISIT 2008]

Multi-session Mutualcast Theorem

NEW!

- Capacity region is given by set of linear inequalities:

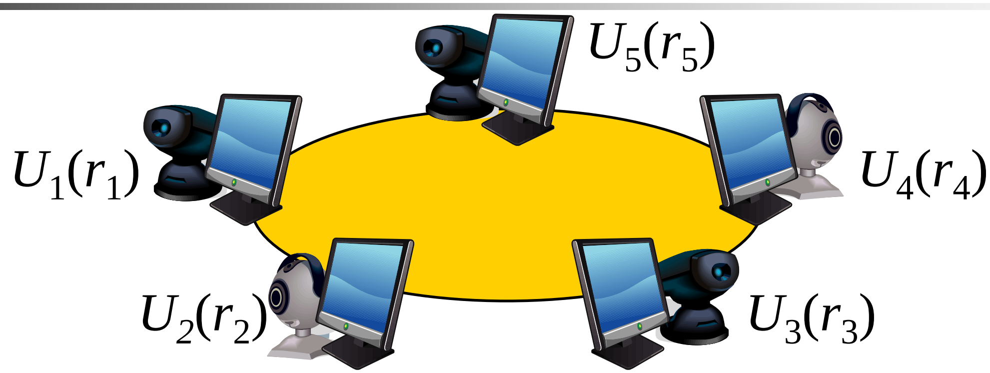
$$x_{i1} + \cdots + x_{i|V|} = r_i, \ i = 1, \ldots, |S|$$

$$\sum_{ij} x_{ij} \deg_{vij} \le c_{out}(v), \ v \in V$$

- Remark:
  - Despite multiple sessions (and helpers), still no inter-session or intra-session network coding is needed!
  - Still $|V|$ trees instead of $O(|V|^{|V|})$; still simple, short

[Sengupta, Chen, Chou, Li; ISIT 2008]

Application to Video Conferencing



- Maximizing $\sum U_i(r_i)$ over all $(r_1, \ldots, r_{|S|}) \in \mathrm{R}$ can be found by maximizing $\sum U_i(r_i)$ over all tree rates x_{ij}, $i=1,\ldots,|S|$, $j=1,\ldots,|V|$, s.t.
  - $x_{i1} + \cdots + x_{i|V|} = r_i\ ,\ i=1,\ldots,|S|$
  - $\sum_{ij} x_{ij} \deg_{ij} \le c_{out}(v),\ v \in V$
- Network Utility Maximization problem

Summary

- For general multi-session multicast over directed graphs with edge capacities
  - Capacity region is stated in terms of Γ^*
  - Hard to know if a rate vector **r** is in capacity region
- For restricted case of multi-session multicast with identical or disjoint receiver sets over P2P networks
  - Capacity region is given by $|V|$ inequalities over at most $|V|\times|S|$ variables (rates on each tree)
  - Any point in this region is achievable by routing over at most $|V|\times|S|$ depth-1 and depth-2 trees